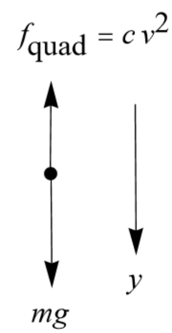
Problem 2.31

A basketball has mass $m = 600$ g and diameter $D = 24$ cm. **(a)** What is its terminal speed? **(b)** If it is dropped from a 30-m tower, how long does it take to hit the ground and how fast is it going when it does so? Compare with the corresponding numbers in a vacuum.

Solution

Draw a free-body diagram for a basketball falling down in a medium with quadratic air resistance. Let the positive y-direction point downward.



Apply Newton's second law in the y-direction.

$$\sum F_y = ma_y$$

Let $v_y = v$ to simplify the notation.

$$mg - cv^2 = m\frac{dv}{dt} \tag{1}$$

The terminal speed occurs when the velocity reaches equilibrium.

$$mg - cv_{\text{ter}}^2 = m(0)$$

Solve for v_{ter}, the terminal velocity.

$$v_{\text{ter}} = \sqrt{\frac{mg}{c}}$$

Part (a)

Since the basketball is spherical, $c = \gamma D^2$. And assuming the air is at STP, $\gamma = 0.25\ \text{N}\cdot\text{s}^2/\text{m}^4$. Therefore,

$$v_{\text{ter}} = \sqrt{\frac{mg}{\gamma D^2}} = \sqrt{\frac{\left(600\ \text{g} \times \frac{1\ \text{kg}}{1000\ \text{g}}\right)\left(9.81\ \frac{\text{m}}{\text{s}^2}\right)}{\left(0.25\ \frac{\text{N}\cdot\text{s}^2}{\text{m}^4}\right)\left(24\ \text{cm} \times \frac{1\ \text{m}}{100\ \text{cm}}\right)^2}} \approx 20.2\ \frac{\text{m}}{\text{s}}.$$

Part (b)

The problem gives a height and asks how long the basketball takes to hit the floor, so the position as a function of time is needed. Once the time is found, it'll be plugged into the velocity function to find how fast it's travelling at impact. Start by separating variables in equation (1) to get v.

$$c\left(\frac{mg}{c} - v^2\right) = m\frac{dv}{dt}$$

$$\frac{c}{m}\,dt = \frac{dv}{\frac{mg}{c} - v^2}$$

Integrate both sides definitely, assuming that at $t = 0$ the basketball has zero velocity.

$$\int_0^t \frac{c}{m}\,dt' = \int_0^v \frac{dv'}{\frac{mg}{c} - v'^2} \tag{2}$$

Make the following substitution in the integral on the right side.

$$v' = \sqrt{\frac{mg}{c}}\sin\theta$$

$$dv' = \sqrt{\frac{mg}{c}}\cos\theta\,d\theta$$

Consequently, equation (2) becomes

$$\frac{c}{m}(t-0) = \int_{\sin^{-1}\left(\frac{0}{\sqrt{\frac{mg}{c}}}\right)}^{\sin^{-1}\left(\frac{v}{\sqrt{\frac{mg}{c}}}\right)} \frac{\sqrt{\frac{mg}{c}}\cos\theta\,d\theta}{\frac{mg}{c}(1-\sin^2\theta)}$$

$$\frac{c}{m}t = \frac{1}{\sqrt{\frac{mg}{c}}}\int_0^{\sin^{-1}\left(\frac{v}{\sqrt{\frac{mg}{c}}}\right)} \frac{\cos\theta\,d\theta}{\cos^2\theta}$$

$$= \sqrt{\frac{c}{mg}}\int_0^{\sin^{-1}\left(\frac{v}{\sqrt{\frac{mg}{c}}}\right)} \sec\theta\,d\theta$$

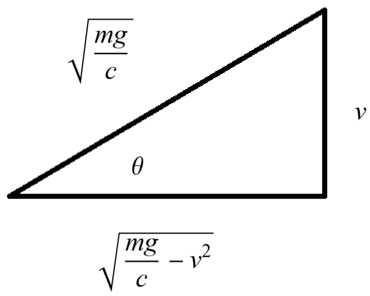
$$= \sqrt{\frac{c}{mg}}\ln|\sec\theta + \tan\theta|\Big|_0^{\sin^{-1}\left(\frac{v}{\sqrt{\frac{mg}{c}}}\right)}$$

$$= \sqrt{\frac{c}{mg}}\ln\left[\frac{\sec\sin^{-1}\left(\frac{v}{\sqrt{\frac{mg}{c}}}\right) + \tan\sin^{-1}\left(\frac{v}{\sqrt{\frac{mg}{c}}}\right)}{\sec 0 + \tan 0}\right].$$

The denominator is $\sec 0 + \tan 0 = 1 + 0 = 1$.

$$\frac{c}{m}t = \sqrt{\frac{c}{mg}} \ln \left[\sec \sin^{-1}\left(\frac{v}{\sqrt{\frac{mg}{c}}}\right) + \tan \sin^{-1}\left(\frac{v}{\sqrt{\frac{mg}{c}}}\right) \right]$$

Draw the implied right triangle.



As a result, the formula simplifies to

$$\begin{aligned}
\frac{c}{m}t &= \sqrt{\frac{c}{mg}} \ln \left(\frac{\sqrt{\frac{mg}{c}}}{\sqrt{\frac{mg}{c} - v^2}} + \frac{v}{\sqrt{\frac{mg}{c} - v^2}} \right) \\
&= \sqrt{\frac{c}{mg}} \ln \left(\frac{\sqrt{\frac{mg}{c}} + v}{\sqrt{\frac{mg}{c} - v^2}} \right) \\
&= \sqrt{\frac{c}{mg}} \ln \left[\frac{\sqrt{\frac{mg}{c}} + v}{\sqrt{\left(\sqrt{\frac{mg}{c}} + v\right)\left(\sqrt{\frac{mg}{c}} - v\right)}} \right] \\
&= \sqrt{\frac{c}{mg}} \ln \sqrt{\frac{\sqrt{\frac{mg}{c}} + v}{\sqrt{\frac{mg}{c}} - v}} \\
&= \sqrt{\frac{c}{mg}} \left[\frac{1}{2} \ln \left(\frac{1 + \frac{v}{\sqrt{\frac{mg}{c}}}}{1 - \frac{v}{\sqrt{\frac{mg}{c}}}} \right) \right] \\
&= \sqrt{\frac{c}{mg}} \tanh^{-1}\left(\frac{v}{\sqrt{\frac{mg}{c}}}\right).
\end{aligned}$$

Solve this equation for v.

$$\tanh^{-1}\left(\frac{v}{\sqrt{\frac{mg}{c}}}\right) = \sqrt{\frac{cg}{m}}\, t$$

As a result, the velocity is

$$\frac{v}{\sqrt{\frac{mg}{c}}} = \tanh\left(\sqrt{\frac{cg}{m}}\,t\right)$$

$$v(t) = \sqrt{\frac{mg}{c}}\tanh\left(\sqrt{\frac{cg}{m}}\,t\right).$$

To get an equation involving position, replace $v(t)$ with dy/dt.

$$\frac{dy}{dt} = \sqrt{\frac{mg}{c}}\tanh\left(\sqrt{\frac{cg}{m}}\,t\right)$$

This is a first-order differential equation for y, so it can be solved by separating variables.

$$dy = \sqrt{\frac{mg}{c}}\tanh\left(\sqrt{\frac{cg}{m}}\,t\right)dt$$

Integrate both sides definitely, assuming that at $t = 0$ the position is $y = 0$ and at some time $t = t_{\max}$ the position is $y = 30$.

$$\int_0^{30} dy = \int_0^{t_{\max}} \sqrt{\frac{mg}{c}}\tanh\left(\sqrt{\frac{cg}{m}}\,t\right)dt$$

$$30 = \sqrt{\frac{mg}{c}}\int_0^{t_{\max}} \tanh\left(\sqrt{\frac{cg}{m}}\,t\right)dt$$

Make the following substitution in the integral on the right.

$$u = \sqrt{\frac{cg}{m}}\,t$$

$$du = \sqrt{\frac{cg}{m}}\,dt \quad \rightarrow \quad \sqrt{\frac{m}{cg}}\,du = dt$$

Consequently,

$$30 = \sqrt{\frac{mg}{c}}\int_0^{\sqrt{\frac{cg}{m}}\,t_{\max}} (\tanh u)\left(\sqrt{\frac{m}{cg}}\,du\right)$$

$$= \frac{m}{c}\int_0^{\sqrt{\frac{cg}{m}}\,t_{\max}} \tanh u\,du$$

$$= \frac{m}{c}\ln(\cosh u)\Big|_0^{\sqrt{\frac{cg}{m}}\,t_{\max}}$$

$$= \frac{m}{c}\ln\left[\frac{\cosh\left(\sqrt{\frac{cg}{m}}\,t_{\max}\right)}{\cosh 0}\right].$$

The denominator is $\cosh 0 = 1$.

$$30 = \frac{m}{c}\ln\left[\cosh\left(\sqrt{\frac{cg}{m}}\,t_{\text{max}}\right)\right]$$

Solve this equation for t_{max}, the time it takes for the basketball to hit the floor.

$$\frac{30c}{m} = \ln\left[\cosh\left(\sqrt{\frac{cg}{m}}\,t_{\text{max}}\right)\right]$$

$$\exp\left(\frac{30c}{m}\right) = \cosh\left(\sqrt{\frac{cg}{m}}\,t_{\text{max}}\right)$$

$$\cosh^{-1}\left[\exp\left(\frac{30c}{m}\right)\right] = \sqrt{\frac{cg}{m}}\,t_{\text{max}}$$

Therefore,

$$t_{\text{max}} = \sqrt{\frac{m}{cg}}\cosh^{-1}\left[\exp\left(\frac{30c}{m}\right)\right].$$

Plugging in the numbers gives

$$t_{\text{max}} = \sqrt{\frac{0.6}{0.25(0.24)^2(9.81)}}\cosh^{-1}\left[\exp\left(\frac{30(0.25)(0.24)^2}{0.6}\right)\right] \approx 2.78 \text{ seconds}$$

for the time to hit the floor. The velocity at impact is

$$\begin{aligned} v(t_{\text{max}}) &= \sqrt{\frac{mg}{c}}\tanh\left(\sqrt{\frac{cg}{m}}\,t_{\text{max}}\right) \\ &\approx \sqrt{\frac{0.6(9.81)}{(0.25)(0.24)^2}}\tanh\left[\sqrt{\frac{(0.25)(0.24)^2(9.81)}{0.6}}\,(2.78)\right] \\ &\approx 17.7\ \frac{\text{m}}{\text{s}}. \end{aligned}$$

In the absence of air resistance (in vacuum), Newton's second law gives a much simpler equation to solve.

$$mg = m\frac{dv}{dt} \quad \to \quad \frac{dv}{dt} = g \quad \to \quad v(t) = gt \quad \to \quad \frac{dy}{dt} = gt \quad \to \quad y = \frac{1}{2}gt^2$$

The time to hit the floor from a height of 30 meters in this case is

$$30 = \frac{1}{2}gt_{\text{max}}^2 \quad \to \quad t_{\text{max}} = \sqrt{\frac{60}{g}} \approx 2.47 \text{ seconds},$$

and the velocity at impact is

$$v(t_{\text{max}}) = gt_{\text{max}} = \sqrt{60g} \approx 24.3\ \frac{\text{m}}{\text{s}}.$$